

Quantum correlations expressed as information and entropic inequalities for composite and noncomposite systems

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Abstract. For noncomposite systems in classical and quantum domains, we obtain new inequalities such as the subadditivity and strong subadditivity conditions for Shannon entropies and information determined by the probability distributions and for von Neumann entropies of quantum states determined by the density operators. We extend the relations of Shannon and Tsallis entropies to the entropies of conditional probability distributions known for composite systems to the case of noncomposite systems. We give a review of the approach to construct the tomographic-probability distributions for qudit systems and present the entropic and information inequalities for spin tomograms, as well as the subadditivity and strong subadditivity conditions for tomograms of the both noncomposite and composite system states.

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1. Introduction

Quantum correlations are the properties of quantum systems which can be associated with different phenomena. The correlations are determined by the structure of the density operator [1, 2]. The density operator of both composite and noncomposite quantum systems has only nonnegative eigenvalues, and this property provides the constraints determining the domain of parameters associated with matrix elements of the state density matrices in arbitrary representations. For composite systems, the density operator has different structure with respect to the decomposition in the form of convex sum of direct products of the density operators of the subsystem states.

The entanglement [3] of composite quantum systems containing two or more subsystems is one of the important manifestations of quantum correlations. The entanglement for two qubits can be detected by violation of the Bell inequality [4, 5].

Recently [6], it was demonstrated that the density operators can be mapped onto the probability vectors (quantum tomographic probabilities called state tomograms), which determine the operators. Recent reviews of this approach can be found in

[7, 8]. Information and entropic properties of quantum systems were studied in the tomographic-probability representation in quantum mechanics in [9–12].

The other property reflecting the presence of quantum correlations in composite systems is quantum discord [13–16]; it corresponds to a difference in information properties of bipartite classical and quantum systems. The discord shows a difference between the properties of the Shannon entropy [17] (determined by the probability distribution describing the classical state of a composite system) and the von Neumann entropy of the quantum bipartite system. Quantum correlations in noncomposite quantum systems were found to be described by the contextuality phenomenon [18–22].

Any probability distribution, in addition to the Shannon entropy, determines the so-called q -entropies [23, 24] containing an extra parameter dependence. The q -entropies are also defined for quantum states, and they are determined by the state density operator.

The Shannon and q -entropies obey some inequalities for composite systems in both classical and quantum domains. The strong subadditivity condition for the von Neumann entropy of three-partite system was proved in [25]. There are other aspects of the entropic inequalities studied in [26–32].

An entropic inequality called the subadditivity condition for the Shannon entropy of the classical bipartite system and the von Neumann entropy of the quantum bipartite system corresponds to the degree of correlations of the subsystem degrees of freedom. Recently [33–35], it was shown that analogous subadditivity conditions and information also exist for noncomposite systems, both classical and quantum. The subadditivity condition and information correspond to the degree of intrinsic correlations between different groups of the results obtained in the experiments where the classical and quantum observables are measured. An analogous result about the existence of the strong subadditivity condition for noncomposite classical and quantum systems was obtained in [35].

The aim of this paper is to review the new entropic and information inequalities found for noncomposite systems following [33–35] and introduce for these systems new concepts – analogs of the conditional entropies associated with composite systems and obtain new equalities characterizing the conditional entropies for Shannon entropy and q -entropies.

This paper is organized as follows.

In section 2, we review the properties of sets of nonnegative numbers organized as tables with different labels by collective indices and obtain various inequalities for these nonnegative numbers. In section 3, we study new entropic inequalities for tomographic entropies of qudit states. In section 4, we present new entropic inequalities for density matrices of noncomposite systems, while in section 5 we discuss the quantum discord and its properties for noncomposite systems. We give our conclusions and prospectives in section 6.

2. Tables of nonnegative numbers

In this section, we describe a tool how to consider any set of N nonnegative numbers, the sum of which is equal to unity, as a set analogous to a vector or a set considered as a matrix. Also we show how such a set can be viewed as a table where each position in the table is labeled by three integers, or four integers, etc. The inverse procedure can also be suggested. This means that an arbitrary table which contains nonnegative numbers can be viewed as a vector with nonnegative number components. This vector can be identified with the probability vector or a point in the simplex. Also the table, where the nonnegative numbers are situated in the positions labeled by some collection of integers, can be treated as a joint probability distribution. When we identify the table with nonnegative numbers with the joint probability distribution, we take into account an additional information arising from the results of experiments where some random variables are measured.

As it was pointed out in [34] and employed in [33, 35], there exist some entropic inequalities which can be associated not only with the probability distributions but with the sets of nonnegative numbers independently on the fact that these numbers are used as probabilities. Also in [35], it was stated that some properties of the von Neumann entropy can be considered as the properties of the complex number set where the complex numbers are associated with matrix elements of the quantum-state density matrix. Thus, in this paper we concentrate on the question: What inequalities are completely mathematical characteristics of sets of either nonnegative numbers or complex numbers, and what inequalities incorporate information that the numbers are associated with physical characteristics of the classical or quantum systems?

Thus, we can consider N nonnegative numbers and label these numbers by the integer number $k = 1, 2, \dots, N$. We obtain a set of nonnegative numbers $\vec{p} = (p_1, p_2, \dots, p_N)$, where $p_k \geq 0$ and $\sum_{k=1}^N p_k = 1$, and the vector notation \vec{p} is introduced. If we add M zero numbers to this set, we get the set of $N' = N + M$ nonnegative numbers; the number M can be chosen arbitrary. This means that one can construct another vector \vec{p}' with N' components. Some of these components are equal to zero.

Let us consider the same initial set of N nonnegative numbers. We can introduce a different labeling of these numbers if, e.g., the chosen integer has the form $N = N_1 N_2$, where N_1 and N_2 are also integers. Then we organize these numbers as a table with labels (jk) . Thus, we obtain \mathcal{P}_{kj} , where $k = 1, 2, \dots, N_1$ and $j = 1, 2, \dots, N_2$, satisfying $\sum_{k=1}^{N_1} \sum_{j=1}^{N_2} \mathcal{P}_{kj} = 1$. The numbers N_1 and N_2 can be chosen in a different way, if the number $N = n_1 n_2 \dots n_s$ with factors n_α ($\alpha = 1, 2, \dots, s$) equal to integers. In fact, we can combine the integers in this product, e.g., as $n_1 = N_1$, $n_2 n_3 \dots n_s = N_2$. Thus, we can map the initial set of N nonnegative numbers on anyone of the tables \mathcal{P}_{jk} constructed according to the described representation of the integer N in the form of the product of the integers. If the number N is prime number, one can add zeros and consider the integer N' for which one has the representation in the form of product of integers. Thus, one can always find many tables \mathcal{P}_{jk} , where indices (jk) play the role

of positions, and the index α labels a particular table associated, e.g., with different numbers N' .

The consideration above presented demonstrates that any set of nonnegative numbers can be mapped onto many tables (matrices \mathcal{P}_{jk}). These rectangular matrices have matrix elements equal either to zero or to a number p_k from the initial vector \vec{p} . One can continue this construction of maps to associate the initial N nonnegative numbers p_k with, e.g., the table Π_{kjl} ($k = 1, 2, \dots, N_1$, $j = 1, 2, \dots, N_2$, and $l = 1, 2, \dots, N_3$). The positions in this table are labeled by three integers. We used the decomposition $N = N_1 N_2 N_3$ of integer N into product of three integers. As it is clear from the previous discussion, such a decomposition can always be found by constructing the integer $N' = N + M$ and using the table containing N' numbers. N numbers in this table Π_{kjl} are the initial N numbers p_k , and the other numbers are zeros.

It is clear that there exist many possibilities to construct different tables $\Pi_{kjl}^{(\alpha)}$ depending on the choice of the integer M and integers N_1 , N_2 , and N_3 providing the decomposition $N' = N_1 N_2 N_3$. The index α labels different tables. It is also clear that for any initial set of nonnegative number p_k one can construct the map of these numbers onto table T_{j_1, j_2, \dots, j_k} containing all these numbers and the corresponding number of zero components. If the nonnegative numbers are experimentally measured probabilities, the tables constructed can be associated with probability distributions. The chosen table \mathcal{P}_{kj} of nonnegative numbers provides two sets \mathcal{P}_{1k} and \mathcal{P}_{2j} of nonnegative numbers

$$P_{1k} = \sum_{j=1}^{N_2} \mathcal{P}_{kj}, \quad P_{2j} = \sum_{k=1}^{N_1} \mathcal{P}_{kj}. \quad (1)$$

Also the chosen table of nonnegative numbers Π_{kjl} provides the three other tables of nonnegative parameters

$$\mathcal{P}_{kj}^{(12)} = \sum_{l=1}^{N_3} \Pi_{kjl}, \quad \mathcal{P}_{jl}^{(23)} = \sum_{k=1}^{N_1} \Pi_{kjl}, \quad P_j^{(2)} = \sum_{k=1}^{N_1} \sum_{l=1}^{N_3} \Pi_{kjl}. \quad (2)$$

If the considered tables of nonnegative numbers are identified with joint probability distributions of physical composite systems, the above formulas correspond to constructing marginal distributions for bipartite and tripartite systems, respectively.

The constructed map of a set of nonnegative numbers p_k onto different kinds of tables, like \mathcal{P}_{kj} or Π_{kjl} , etc., has an inverse in the following sense.

For a given table with $N_1 N_2$ positions (matrix \mathcal{P}_{kj}), one can construct N -vector using the map of pairs of integers onto integers as $(11) \rightarrow 1, (12) \rightarrow 2, \dots, (1N_2) \rightarrow N_2, (2, 1) \rightarrow N_2 + 1, (2, 2) \rightarrow N_2 + 2, \dots, (N_1 N_2) \rightarrow N$. Analogously, the table of nonnegative numbers Π_{kjl} can be mapped onto the N -vector. This observation can be used to rewrite the relations available for tables \mathcal{P}_{kj} or Π_{kjl} in terms of numbers p_k . For example, it is known that joint probability distributions provide inequalities called the subadditivity conditions and strong subadditivity conditions, and these inequalities are valid for matrices \mathcal{P}_{kj} and joint probability distributions Π_{kjl} , respectively. Being rewritten in the form of probability vector components p_k , they are valid for an arbitrary set of nonnegative numbers, which are equal to the probabilities.

Example of $N = 7$

For $N = 7$, one has the inequalities [35]

$$\begin{aligned}
& \left(- \sum_{k=1}^7 p_k \ln p_k \right) - (p_1 + p_2 + p_5 + p_6) \ln(p_1 + p_2 + p_5 + p_6) \\
& - (p_3 + p_4 + p_7) \ln(p_3 + p_4 + p_7) \leq - (p_1 + p_2) \ln(p_1 + p_2) - (p_3 + p_4) \ln(p_3 + p_4) \\
& - (p_5 + p_6) \ln(p_5 + p_6) - p_7 \ln p_7 - (p_1 + p_5) \ln(p_1 + p_5) - (p_2 + p_6) \ln(p_2 + p_6) \\
& - (p_3 + p_7) \ln(p_3 + p_7) - p_4 \ln p_4
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
& - \sum_{k=1}^7 p_k \ln p_k \leq - (p_1 + p_2 + p_5 + p_6) \ln(p_1 + p_2 + p_5 + p_6) \\
& - (p_3 + p_4 + p_7) \ln(p_3 + p_4 + p_7) - (p_1 + p_3) \ln(p_1 + p_3) \\
& - (p_2 + p_4) \ln(p_2 + p_4) - (p_5 + p_7) \ln(p_5 + p_7) - p_4 \ln p_4.
\end{aligned} \tag{4}$$

These inequalities are analogs of the strong subadditivity condition for three-partite system and the subadditivity condition for two-partite system, respectively. In spite of the fact that the 7 nonnegative numbers are not associated with any probabilities, the inequalities are satisfied. The inequalities were obtained by introducing an N' -dimensional vector with $N' = N + 1$ with the component $p_8 = 0$. The eight-dimensional vector obtained was mapped onto the rectangular matrix \mathcal{P}_{kj} corresponding to the decomposition $N' = 2 \cdot 4$, and this provided the subadditivity condition (4). The strong subadditivity condition (3) corresponds to the decomposition $N' = 2 \cdot 2 \cdot 2$.

For four nonnegative numbers $\vec{p} = (a, b, c, d)$, an analog of the subadditivity condition discussed in [34] reads

$$\begin{aligned}
& - (a + b) \ln(a + b) - (c + d) \ln(c + d) - (a + c) \ln(a + c) - (b + d) \ln(b + d) \\
& \geq -a \ln a - b \ln b - c \ln c - d \ln d.
\end{aligned} \tag{5}$$

Inequalities for tables

The inequalities above discussed are examples of general inequalities for matrix tables \mathcal{P}_{kj} and Π_{kjl} containing the nonnegative numbers; they are

$$\begin{aligned}
& - \sum_{j=1}^m \left(\sum_{k=1}^n \mathcal{P}_{jk} \right) \left(\ln \sum_{k'=1}^n \mathcal{P}_{jk'} \right) - \sum_{k=1}^n \left(\sum_{j=1}^m \mathcal{P}_{jk} \right) \left(\ln \sum_{j'=1}^m \mathcal{P}_{jj'} \right) \\
& \geq - \sum_{j=1}^m \sum_{k=1}^n \mathcal{P}_{jk} \ln \mathcal{P}_{jk},
\end{aligned} \tag{6}$$

$$\begin{aligned}
& - \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{m=1}^{n_3} \Pi_{jkm} \ln \Pi_{jkm} - \sum_{k=1}^{n_2} \left(\sum_{j=1}^{n_1} \sum_{m=1}^{n_3} \Pi_{jkm} \right) \ln \left(\sum_{j'=1}^{n_1} \sum_{m'=1}^{n_3} \Pi_{j'km} \right) \\
& \leq - \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \left(\sum_{m'=1}^{n_3} \Pi_{jk m'} \right) \ln \left(\sum_{m=1}^{n_3} \Pi_{jkm} \right) - \sum_{k=1}^{n_2} \sum_{m=1}^{n_3} \left(\sum_{j'=1}^{n_1} \Pi_{j'km} \right) \ln \left(\sum_{j=1}^{n_1} \Pi_{jkm} \right).
\end{aligned} \tag{7}$$

In the case where \mathcal{P}_{kj} is the joint probability distribution for two random variables of the bipartite system, equation (6) is the usual subadditivity condition for Shannon entropies associated with the probability distribution. Inequality (7) is the strong subadditivity condition, if Π_{jkm} is the joint probability distribution of three random variables for the tripartite system. One should stress that the inequalities discussed are valid for an arbitrary probability vector \vec{p} even of the system which is not composite. Such a system can have only a single random variable; nevertheless, there exist the inequalities for the probability distribution of this random variable identical to the subadditivity and strong subadditivity conditions. For the systems, which do not have subsystems, these inequalities can be interpreted as entropic properties corresponding to the existence of correlations between different results of measurements of the random variable.

3. Inequalities for tomographic entropies

For qudit states with the density matrix ρ , the tomographic-probability distribution (spin tomogram) $w(m, \vec{n})$ can be used as an alternative to the density matrix ρ [36, 37]. It is interpreted as the probability to obtain the spin projection m , where $m = -j, -j+1, \dots, j$, on the quantization axes determined by the unit vector $\vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. The tomogram can be obtained as a result of the experiment. Thus, for qudit states, nonnegative numbers p_k can be identified with the tomographic probabilities. For example, for the qudit state corresponding to spin $j = 3/2$ we have the four-vector $\vec{p} = (p_1, p_2, p_3, p_4)$, where $p_1 = w(-3/2, \vec{n})$, $p_2 = w(-1/2, \vec{n})$, $p_3 = w(1/2, \vec{n})$, and $p_4 = w(3/2, \vec{n})$.

The tomographic Shannon entropy is defined as (see, e.g., [38])

$$H(\vec{n}) = - \sum_{m=-3/2}^{3/2} w(m, \vec{n}) \ln w(m, \vec{n}). \quad (8)$$

There are two tomographic-probability distributions

$$W_1(-2, \vec{n}) = w(-3/2, \vec{n}) + w(-1/2, \vec{n}), \quad W_1(2, \vec{n}) = w(1/2, \vec{n}) + w(3/2, \vec{n}), \quad (9)$$

$$W_2(-1, \vec{n}) = w(-3/2, \vec{n}) + w(1/2, \vec{n}), \quad W_2(1, \vec{n}) = w(-1/2, \vec{n}) + w(3/2, \vec{n}), \quad (10)$$

which are analogs of marginal probability distributions for the bipartite system. We can interpret these probability distributions as follows: $W_1(\vec{n})$ yields the probabilities to find the qudit state with the sum of spin projections equal to ± 2 , and $W_2(\vec{n})$ provides the probabilities to find the qudit state with the sum of spin projections equal to ± 1 . In spite the fact that the system is not bipartite, we have the subadditivity condition

$$\begin{aligned} - \sum_{m=-3/2}^{3/2} w(m, \vec{n}) \ln w(m, \vec{n}) &\leq -W_1(1, \vec{n}) \ln W_1(1, \vec{n}) - W_1(-1, \vec{n}) \ln W_1(-1, \vec{n}) \\ &\quad - W_2(2, \vec{n}) \ln W_2(2, \vec{n}) - W_2(-2, \vec{n}) \ln W_2(-2, \vec{n}). \end{aligned} \quad (11)$$

Conditional tomographic probabilities

We introduce the conditional tomographic probability distribution for the qudit state following the procedure of constructing the distribution for the bipartite system. For the qudit-state tomogram with $j = 3/2$, we introduce two conditional probabilities

$$V_1(\vec{n}) = \frac{w(-3/2, \vec{n})}{w(-3/2, \vec{n}) + w(-1/2, \vec{n})}, \quad V_2(\vec{n}) = \frac{w(-1/2, \vec{n})}{w(-3/2, \vec{n}) + w(-1/2, \vec{n})}, \quad (12)$$

$$\tilde{V}_1(\vec{n}) = \frac{w(1/2, \vec{n})}{w(3/2, \vec{n}) + w(1/2, \vec{n})}, \quad \tilde{V}_2(\vec{n}) = \frac{w(3/2, \vec{n})}{w(3/2, \vec{n}) + w(1/2, \vec{n})}. \quad (13)$$

The tomographic conditional probability distributions (12) and (13) determine the tomographic conditional entropies. Since there exists the identity for four nonnegative numbers p_1, p_2, p_3 , and p_4 (considered as the probabilities) of the form

$$\begin{aligned} & -p_1 \ln \frac{p_1}{p_1 + p_2} - p_2 \ln \frac{p_2}{p_1 + p_2} - p_3 \ln \frac{p_3}{p_3 + p_4} - p_4 \ln \frac{p_4}{p_3 + p_4} \\ &= (p_1 + p_2) \ln(p_1 + p_2) + (p_3 + p_4) \ln(p_3 + p_4) - p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4, \end{aligned} \quad (14)$$

one can rewrite this equality as the property of the conditional tomographic entropy

$$H(V, \tilde{V}) = H(V | \tilde{V}) + H(\tilde{V}), \quad (15)$$

where the tomographic entropy of the qudit state reads

$$H(V, \tilde{V}) = - \sum_{m=-3/2}^{3/2} w(m, \vec{n}) \ln w(m, \vec{n}). \quad (16)$$

Formula (15) can be generalized to yield an analog of the chain form of the conditional entropic equality for multipartite system.

The conditional tomographic entropy of the qudit state is defined as

$$H(V | \tilde{V}) = H(V, \tilde{V}) + W_1(2, \vec{n}) \ln W_1(2, \vec{n}) + W_1(-2, \vec{n}) \ln W_1(-2, \vec{n}). \quad (17)$$

Conditional q -entropies

We define the q -entropy for four nonnegative numbers p_1, p_2, p_3 , and p_4 as follows:

$$T_q(\vec{p}) = \frac{1}{1-q} \left(\sum_{k=1}^4 p_k^q - 1 \right). \quad (18)$$

Identifying the numbers p_k with tomographic probabilities of qudit state with $j = 3/2$, we obtain the Tsallis tomographic entropy [38]

$$T_q(\vec{n}) = \frac{1}{1-q} \left(\sum_{m=-3/2}^{3/2} [w(m, \vec{n})]^q - 1 \right). \quad (19)$$

For $q \rightarrow 1$, this entropy tends to the conditional Shannon entropy $H(V | \tilde{V})$.

One can introduce the conditional tomographic q -entropy $T_q(V | \tilde{V})$ for the qudit state with $j = 3/2$, using the formula presented in [34] for bipartite qudit states,

$$T_q(V | \tilde{V}) = T_q(\vec{n}) - \frac{1}{1-q} \left(W_1(-2, \vec{n})^q + W_1(2, \vec{n})^q - 1 \right). \quad (20)$$

Formula (20) can be generalized to yield the chain form of q -entropic equality for the conditional entropies associated with multipartite systems.

In view of the inequalities derived for tomographic q -entropies of bipartite qudit systems in [34], we obtain analogous inequalities for q -entropy of the qudit state under consideration

$$T_q(V) \leq T_q(\vec{n}), \quad T_q(V | \tilde{V}) \leq T_q(V), \quad (21)$$

where

$$T_q(V) = \frac{1}{1-q} \left(W_1(-2, \vec{n})^q + W_1(2, \vec{n})^q - 1 \right), \quad T_q(\tilde{V}) = \frac{1}{1-q} \left(W_2(1, \vec{n})^q + W_2(-1, \vec{n})^q - 1 \right).$$

4. Inequalities for density matrices

The quantum states of qudits with spin j ($2j+1 = N$) are determined by density $N \times N$ -matrices ρ_{km} . We can apply our tool elaborated for the set of complex numbers ρ_{km} . In this case, the tool is to map a pair of integers (k, m) onto the other collective pairs of integers, e.g., $k \rightarrow (l_k, j_k)$ and $m \rightarrow (l_m, j_m)$, i.e., $(k, m) \rightarrow (l_k j_k, l_m j_m)$. Continuing this procedure, we construct the map $(k, m) \rightarrow (l_k j_k n_k, l_m j_m n_m)$, etc. This means that the density matrix $\rho_{mm'}$ with $j = 3/2$ can be written in the form

$$\rho = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{11,21} & \rho_{11,22} \\ \rho_{12,11} & \rho_{12,12} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{21,21} & \rho_{21,22} \\ \rho_{22,11} & \rho_{22,12} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}. \quad (22)$$

Such a form provides a possibility to apply to this density matrix the positive maps which yield the nonnegative matrices

$$\rho(1)_{jk} = \sum_{s=1}^2 \rho_{js,ks}, \quad \rho(2)_{mn} = \sum_{s=1}^2 \rho_{sm,sn}. \quad (23)$$

Complex numbers $\rho_{mm'}$ satisfy the criterion of nonnegativity of the matrix ρ , as well as the complex numbers $\rho(1)_{jk}$ and $\rho(2)_{mn}$. Then we have the subadditivity condition

$$-\text{Tr } \rho \ln \rho \leq -\text{Tr } \rho(1) \ln \rho(1) - \text{Tr } \rho(2) \ln \rho(2). \quad (24)$$

This inequality is valid for one qudit state, and the qudit is a system which does not contain subsystems.

Examples of spin-2 and spin-3 states

There are other examples [34] of such inequalities which are analogs of the strong subadditivity condition for qudit states with $j = 2$ and $j = 3$. In the case of 7×7 -matrix ρ_{jk} , it has the form

$$-\text{Tr } (\rho \ln \rho) - \text{Tr } (R_2 \ln R_2) \leq -\text{Tr } (R_{12} \ln R_{12}) - \text{Tr } (R_{23} \ln R_{23}), \quad (25)$$

where the density matrix R_{12} has matrix elements expressed in terms of the density matrix ρ_{jk} as follows:

$$R_{12} = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} + \rho_{26} & \rho_{17} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} + \rho_{46} & \rho_{37} \\ \rho_{51} + \rho_{62} & \rho_{53} + \rho_{64} & \rho_{55} + \rho_{66} & \rho_{57} \\ \rho_{71} & \rho_{73} & \rho_{75} & \rho_{77} \end{pmatrix}. \quad (26)$$

The density matrix R_{23} reads

$$R_{23} = \begin{pmatrix} \rho_{11} + \rho_{55} & \rho_{12} + \rho_{56} & \rho_{13} + \rho_{57} & \rho_{14} \\ \rho_{21} + \rho_{65} & \rho_{22} + \rho_{66} & \rho_{23} + \rho_{67} & \rho_{24} \\ \rho_{31} + \rho_{75} & \rho_{32} + \rho_{76} & \rho_{33} + \rho_{77} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \quad (27)$$

while the matrix R_2 is

$$R_2 = \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{55} + \rho_{66} & \rho_{13} + \rho_{24} + \rho_{57} \\ \rho_{31} + \rho_{42} + \rho_{75} & \rho_{33} + \rho_{44} + \rho_{77} \end{pmatrix}. \quad (28)$$

In the case of 5×5 -matrix ρ_{lk} corresponding to the qudit state with $j = 2$, we have inequality (25), where

$$R_{12} = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} \\ \rho_{51} & \rho_{53} & \rho_{55} \end{pmatrix}, \quad R_{23} = \begin{pmatrix} \rho_{11} + \rho_{55} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \quad (29)$$

$$R_2 = \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{55} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}. \quad (30)$$

In order to obtain these inequalities, we mapped the matrices ρ_{lk} on matrices R_{lk} acting on vectors in the eight-dimensional linear space. The matrix R_{lk} has matrix elements of the matrix ρ , and the other matrix elements are zeros.

The quantum subadditivity condition for qutrit state was obtained in [33]. The procedure developed is analogous to mapping of N -vectors on N' -vectors with adding zero components to the initial vector. After this map, we apply the known subadditivity and strong subadditivity conditions to the obtained nonnegative matrix R_{lk} . As a result, we get the above entropic inequalities for the density matrices of qudit states. It is obvious that analogous generalized inequalities can be obtained for arbitrary qudits with $j = 3/2, 2, 5/2, 3, 7/2, \dots$. Also the strong subadditivity condition, which is usually associated with three-partite systems, can be written for the two-qudit system, in view of the suggested map of integers on the collection of integers.

5. Some inequalities and quantum discord

For a bipartite system with the joint probability distribution \mathcal{P}_{jk} , the Shannon information is defined as a difference of the sum of subsystem entropies and entropy of the composite system. The two-qudit state can be described by the joint tomographic

probability distribution $w(m_1, m_2, u)$, which is given by diagonal matrix elements of the density matrix $\rho(1, 2)$ in unitary rotated basis

$$w(m_1, m_2, u) = \langle m_1 m_2 | u^\dagger \rho(1, 2) u | m_1 m_2 \rangle, \quad (31)$$

where u is the unitary matrix.

In the case $u = u_1 \otimes u_2$, where u_1 and u_2 are matrices of the irreducible representation of the rotation group, the tomogram $w(m_1, m_2, \vec{n}_1, \vec{n}_2)$ is the joint probability distribution of two random spin projections m_1 and m_2 . The directions \vec{n}_1 and \vec{n}_2 are determined by Euler angles which fix the matrix elements of the representation matrices u_1 and u_2 . The Shannon entropy

$$H_{12}(u) = - \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} w(m_1, m_2, u) \ln w(m_1, m_2, u) \quad (32)$$

has the minimum for a specific value of the unitary matrix u [9], which is equal to the von Neumann entropy associated with the density matrix of the two qudit states, i.e.,

$$\min H(u) = -\text{Tr} \rho(1, 2) \ln \rho(1, 2). \quad (33)$$

The density matrices $\rho(1) = -\text{Tr}_2 \rho(1, 2)$ and $\rho(2) = -\text{Tr}_1 \rho(1, 2)$ of the qudits give the von Neumann entropies $S(1) = -\text{Tr} \rho(1) \ln \rho(1)$ and $S(2) = -\text{Tr} \rho(2) \ln \rho(2)$. These entropies can be found as minima of the Shannon entropies

$$H_k(u) = - \sum_{m_k=-j_k}^{j_k} w_k(m_k, u) \ln w_k(m_k, u), \quad k = 1, 2 \quad (34)$$

with respect to local unitary transforms $u = u_{10} \otimes u_{20}$. The subadditivity condition reads

$$S_{12} \leq S_1 + S_2. \quad (35)$$

In [34], we showed that inequality (35) can be written as

$$S_1 + S_2 \geq H_{12}(u_{10} \otimes u_{20}) \geq S_{12}. \quad (36)$$

The tomographic discord introduced as the difference

$$\mathcal{D} = (S_1 + S_2 - S_{12}) - I(u_{10} \otimes u_{20}) \geq 0 \quad (37)$$

characterizes quantum correlations in the bipartite system of qudits.

Quantum discord for spin-3/2 state

In view of the inequalities discussed above, we can introduce the quantum discord for one qudit state. For example, the qudit state corresponding to $j = 3/2$ with density matrix (22) determines the state tomogram $w(m, u) = \langle m | u \rho u^\dagger | m \rangle$, $m = -3/2, -1/2, 1/2, 3/2$. The von Neumann entropy of this state is the minimum of the Shannon entropy determined by the tomographic probability distribution; this means that $\min_u \left(- \sum_{m=-3/2}^{3/2} w(m, u) \right) = -\text{Tr} \rho \ln \rho$. We rewrite the matrix in two

forms: one form corresponds to an arbitrary matrix, and the other one specifies the spin-3/2 state,

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \equiv \begin{pmatrix} \rho_{-3/2} -3/2 & \rho_{-3/2} -1/2 & \rho_{-3/2} 1/2 & \rho_{-3/2} 3/2 \\ \rho_{-1/2} -3/2 & \rho_{-1/2} -1/2 & \rho_{-1/2} 1/2 & \rho_{-1/2} 3/2 \\ \rho_{1/2} -3/2 & \rho_{1/2} -1/2 & \rho_{1/2} 1/2 & \rho_{1/2} 3/2 \\ \rho_{3/2} -3/2 & \rho_{3/2} -1/2 & \rho_{3/2} 1/2 & \rho_{3/2} 3/2 \end{pmatrix}. \quad (38)$$

We consider two density matrices $\rho(1)$ and $\rho(2)$ by taking analogs of tracing with respect to subsystem degrees of freedom; the 2×2 -matrices read

$$\rho(1) = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}. \quad (39)$$

We constructed the 2×2 -matrices from arbitrary density 4×4 -matrices. In notation relevant to the spin-3/2 state, these density matrices read

$$\begin{aligned} \rho^{(3/2)}(1) &= \begin{pmatrix} \rho_{-3/2} -3/2 + \rho_{-1/2} -1/2 & \rho_{-3/2} 1/2 + \rho_{-1/2} 3/2 \\ \rho_{-1/2} -3/2 + \rho_{-3/2} -1/2 & \rho_{1/2} 1/2 + \rho_{3/2} 3/2 \end{pmatrix}, \\ \rho^{(3/2)}(2) &= \begin{pmatrix} \rho_{-3/2} -3/2 + \rho_{1/2} -1/2 & \rho_{-3/2} -1/2 + \rho_{-1/2} 3/2 \\ \rho_{-1/2} -3/2 + \rho_{3/2} -1/2 & \rho_{-1/2} -1/2 + \rho_{3/2} 3/2 \end{pmatrix}; \end{aligned} \quad (40)$$

they determine the tomograms. While deriving (40), we used the other notation for the matrices $\rho^{(3/2)}(1) = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ and $\rho^{(3/2)}(2) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$. The two tomograms $w(\alpha, u_1)$ and $w(\beta, u_2)$ are diagonal matrix elements of the matrices $u_1 \rho^{(3/2)}(1) u_1^\dagger$ and $u_2 \rho^{(3/2)}(2) u_2^\dagger$, they are

$$w(\alpha, u_1) = \langle \alpha | u_1 \rho^{(3/2)}(1) u_1^\dagger | \alpha \rangle, \quad w(\beta, u_2) = \langle \beta | u_2 \rho^{(3/2)}(2) u_2^\dagger | \beta \rangle. \quad (41)$$

The random variables α and β take two values ± 1 . The von Neumann entropies of the “qubit” states with density matrices $\rho^{(3/2)}(1)$ and $\rho^{(3/2)}(2)$ read

$$S^{(3/2)}(1) = -\text{Tr} \rho^{(3/2)}(1) \ln \rho^{(3/2)}(1), \quad S^{(3/2)}(2) = -\text{Tr} \rho^{(3/2)}(2) \ln \rho^{(3/2)}(2). \quad (42)$$

We introduce quantum discord as follows:

$$D = \text{Tr}(\rho \ln \rho) + S^{(3/2)}(1) + S^{(3/2)}(2) - I(u_{10} \times u_{20}), \quad (43)$$

where the tomographic information $I(u)$ is

$$\begin{aligned} I(u) &= \sum_{m=-3/2}^{3/2} \langle m | u \rho u^\dagger | u | \ln \langle m | u \rho u^\dagger | m \rangle \\ &\quad - \sum_{\alpha} \langle \alpha | u_1 \rho^{3/2}(1) u_1^\dagger | \alpha \rangle \ln \left(\sum_{\alpha} \langle \alpha | u_1 \rho^{3/2}(1) u_1^\dagger | \alpha \rangle \right) \\ &\quad - \sum_{\beta} \langle \alpha | u_2 \rho^{3/2}(2) u_2^\dagger | \beta \rangle \ln \left(\sum_{\beta} \langle \beta | u_2 \rho^{3/2}(2) u_2^\dagger | \beta \rangle \right). \end{aligned} \quad (44)$$

One has the inequality

$$\begin{aligned} &S^{(3/2)}(1) + S^{(3/2)}(2) \geq \\ &\quad - \sum_{m=-3/2}^{3/2} \langle m | u_{10} \otimes u_{20} \rho u_{10}^\dagger \otimes u_{20}^\dagger | m \rangle \ln \langle m | u_{10} \otimes u_{20} \rho u_{10}^\dagger \otimes u_{20}^\dagger | m \rangle \\ &\geq -\text{Tr} \rho \ln \rho. \end{aligned} \quad (45)$$

This inequality is a new relation for the density matrix of the spin-3/2 state. In view of inequality (45), the quantum discord is nonnegative $D \geq 0$. The relations obtained are valid for an arbitrary density 4×4 -matrix.

Example of qutrit state

Now we show new inequalities for qutrit state with the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix}, \quad (46)$$

considering it as a particular 4×4 -matrix ρ [33] with matrix elements in the fourth row and column equal to zero. In this case, the 2×2 -matrices associated with the 4×4 -matrices are

$$\rho^{(1)}(1) = \begin{pmatrix} \rho_{11} + \rho_{00} & \rho_{10} \\ \rho_{01} & \rho_{-1-1} \end{pmatrix}, \quad \rho^{(1)}(2) = \begin{pmatrix} \rho_{11} + \rho_{-1-1} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix}, \quad (47)$$

where we used index (1) to point out that the matrix is obtained for $j = 1$.

We have von Neumann entropies $S = -\text{Tr} \rho \ln \rho = -\text{Tr} \tilde{\rho} \ln \tilde{\rho}$, $S_1^{(1)} = -\text{Tr} \rho_1^{(1)} \ln \rho_1^{(1)}$, and $S_2^{(1)} = -\text{Tr} \rho_2^{(1)} \ln \rho_2^{(1)}$. The inequality $S \leq S_1^{(1)} + S_2^{(1)}$ was obtained for the density matrix of the qutrit state in [33], which is a new analog of the subadditivity condition. If we introduce tomographic probabilities corresponding to the matrix $\tilde{\rho}$ as diagonal elements of the matrix $u\tilde{\rho}u^\dagger$, i.e., $w(\alpha, u) = \langle \alpha | u\tilde{\rho}u^\dagger | \alpha \rangle$ and find the unitary 2×2 -matrices u_{10} and u_{20} diagonalizing the matrices $\rho^{(1)}(1)$ and $\rho^{(1)}(2)$, we arrive at a stronger inequality for the qutrit density matrix, namely,

$$S_1^{(1)} + S_2^{(1)} \geq H(u_{10} \otimes u_{20}) \geq S, \quad (48)$$

where the tomographic entropy $H(u)$ for $u = u_{10} \otimes u_{20}$ is

$$H(u_{10} \otimes u_{20}) = - \sum_{\alpha} \langle \alpha | u_{10} \otimes u_{20} \tilde{\rho} u_{10}^\dagger \otimes u_{20}^\dagger | \alpha \rangle \ln \langle \alpha | u_{10} \otimes u_{20} \tilde{\rho} u_{10}^\dagger \otimes u_{20}^\dagger | \alpha \rangle. \quad (49)$$

The quantum discord for the qutrit state reads

$$D^{(1)} = S_1^{(1)} + S_2^{(1)} - S - I(u_{10} \otimes u_{20}). \quad (50)$$

The tomographic information $I(u_{10} \otimes u_{20})$ is determined by the equality for an arbitrary 4×4 -matrix analogously to the case of $j = 3/2$. The discord for qutrit state $D^{(1)}$ is a nonnegative number. This property is a new characteristics of quantum correlations for qutrit states. It is clear that the procedure presented can be used to introduce quantum discord and new inequalities for arbitrary qudit states, as well as for multiqutrit states.

6. Conclusions

To conclude, we point out the main results of our study.

We elaborated the method to extend all entropic and information equalities and inequalities known for composite (both classical and quantum) systems to the case of

noncomposite and composite systems. The method is based on using an invertible map of N integers onto the pairs, triples, etc. of the integers. A particular case of entropic inequalities like the subadditivity condition and the strong subadditivity condition introduced for a single qudit state was studied. We showed examples of these inequalities for qudit states with $j = 2$ and 3 ; the subadditivity condition for qutrit state [33] was also presented.

We introduced the conditional probabilities and entropies for noncomposite qudit systems and constructed chain equalities for Shannon entropy and q -entropies for the qudit. Also the notion of quantum discord was given for a single qudit. We pointed out that the physical meaning of the new information and entropic equalities for noncomposite systems, we derived, needs clarification though it seems that the relations, we found, correspond to intrinsic correlations in the system, even if the system does not have the structure of separated subsystems. These aspects of information properties of noncomposite systems will be considered in a future publication.

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